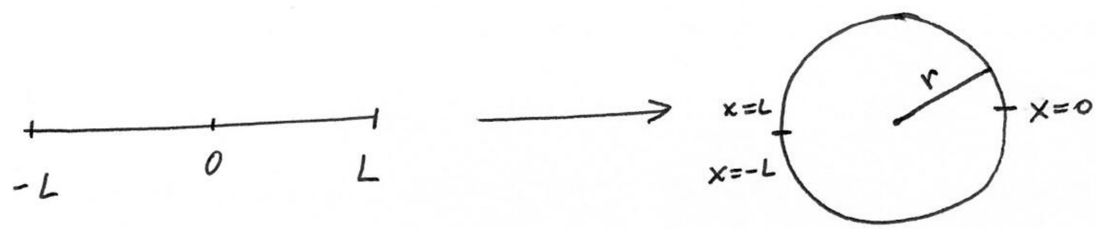
Heat conduction in a thin circular ring.



perimeter $= 2\pi r = 2L$

$\Rightarrow r = L/\pi$.

Assumption:

1) Temperature is constant per cross section.
2) Homogeneous material, constant K_0.
3) Perfect thermal contact at $x = L$ or $x = -L$.
4) No sources of thermal energy.

IBVP:

$$\begin{cases} u_t = k u_{xx}, \quad -L < x < L, \quad t > 0 & (1.1) \\ u(-L,t) = u(L,t) & (1.2) \\ -K_0 u_x(-L,t) = \phi(-L,t) = \phi(L,t) = -K_0 u_x(L,t). & (1.3) \\ u(x,0) = f(x) & (1.4) \end{cases}$$

Using separation of variables:

$$u(x,t) = \phi(x) G(t) \qquad (1.5)$$

We are led to :

Two ODE'S. In fact, replacing (1.5) into (1.1)

$$u_t = k\, u_{xx}$$

$$\phi(x)\, G'(t) = k\, \phi''(x)\, G(t)$$

Separating Variables: Divide by $k\, \phi(x)\, G(t)$

$$\frac{G'(t)}{k\, G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda$$

λ constant

$-$ for convenience.

Since "t" and "x" are completely unrelated.

Then, we arrive to

$$G'(t) + \lambda k\, G(t) = 0 \quad \text{or}$$

$$\boxed{G'(t) = -\lambda k\, G(t)} \qquad (2.1)$$

and

$$\boxed{\phi''(x) + \lambda\, \phi(x) = 0} \qquad (2.2)$$

The B.Cs imply

$$u(-L,t) = u(L,t)$$

$$\phi(-L)\, \cancel{G(t)} = \phi(L)\, \cancel{G(t)}$$

$$\boxed{\phi(-L) = \phi(L)}$$

If $G(t) \equiv 0$ Trivial Soln.

$$u_x(-L,t) = u_x(L,t)$$

$$\phi'(-L)\, \cancel{G(t)} = \phi'(L)\, \cancel{G(t)}$$

$$\boxed{\phi'(-L) = \phi'(L)}$$

Eingenvalue Problem

$$\begin{cases} \phi''(x) + \lambda\, \phi(x) = 0 \\ \phi(-L) = \phi(L) \\ \phi'(-L) = \phi'(L) \end{cases}$$

$\underline{\underline{\lambda > 0}}$ Gen. Soln. : $\phi(x) = C \cos(\sqrt{\lambda} x) + d \sin(\sqrt{\lambda} x)$

$$\phi'(x) = -\sqrt{\lambda}\, C \sin(\sqrt{\lambda} x) + \sqrt{\lambda}\, d \cos(\sqrt{\lambda} x)$$

(I) B.C. $\boxed{\phi(L) = \phi(-L)}$

$$\Leftrightarrow \cancel{C \cos(\sqrt{\lambda} L)} + d \sin(\sqrt{\lambda} L) = \cancel{C \cos(-\sqrt{\lambda} L)} + d \sin(-\sqrt{\lambda} L) \quad (1)$$

(II) B. C. $\boxed{\phi'(L) = \phi'(-L)}$

$$\Leftrightarrow -\sqrt{\lambda}\, C \sin(\sqrt{\lambda} L) + \cancel{\sqrt{\lambda}\, d \cos(\sqrt{\lambda} L)} = -\sqrt{\lambda}\, C \sin(-\sqrt{\lambda} L) + \cancel{\sqrt{\lambda}\, d \cos(-\sqrt{\lambda} L)} \quad (2)$$

From (1) $\quad 2d \sin(\sqrt{\lambda} L) = 0$.

From (2) $\quad 2\sqrt{\lambda}\, C \sin(\sqrt{\lambda} L) = 0$

If $\sin(\sqrt{\lambda} L) \neq 0$, then both $C = 0$ and $d = 0 \Rightarrow$ Trivial Soln. not eigenfn.

then, $\quad \sin(\sqrt{\lambda} L) = 0 \Rightarrow \sqrt{\lambda} L = n\pi \Rightarrow \lambda = \left(\frac{n\pi}{L}\right)^2. \; n = 1, 2, \ldots$

then, eigenfns are

$$\boxed{\phi_n(x) = C_n \cos\left(\tfrac{n\pi}{L} x\right) + d_n \sin\left(\tfrac{n\pi}{L} x\right)}$$

They satisfy the BC's for any C_n and d_n since

$$\phi_n(L) = C_n \cos(n\pi) + d_n \overset{0}{\cancel{\sin(n\pi)}} = C_n \cos(n\pi) = C_n \cos(-n\pi)$$
$$= \phi(-L)$$

Also, $\phi_n'(-L) = -\frac{n\pi}{L} c_n \cancel{\sin(n\pi)}^{0} + \frac{n\pi}{L} d_n \cos(n\pi) =$

$$= \frac{n\pi}{L} d_n \cos(n\pi) = \frac{n\pi}{L} d_n \cos(-n\pi) = \phi_n'(-L), \text{ for all } n$$

So, there are no additional conditions on c_n and d_n.

Therefore, for $\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2,}$ $n = 1, 2, \ldots$ eigenvalues

we have two eigenfns: linearly indep. $\boxed{\phi_n(x) = \begin{cases} \sin\left(\frac{n\pi}{L}x\right) \\ \cos\left(\frac{n\pi}{L}x\right) \end{cases}}$

If $\underline{\underline{\lambda = 0}}$ $\Rightarrow$ $\phi_0'' = 0$ $\Rightarrow$ $\phi_0(x) = c_1 x + c_2$

$$\phi_0(L) = \phi_0(-L) \Leftrightarrow c_1 L + \cancel{c_2} = -c_1 L + \cancel{c_2} \Rightarrow c_1 = -c_1$$

$$\phi_0'(L) = \phi_0'(-L) \Leftrightarrow c_1 = c_1$$

These two equations leads to $c_1 = 0$

$\Rightarrow$ $\boxed{\phi_0(x) \equiv 1}$ is also an eigenfn for $\boxed{\lambda_0 = 0}$.

Back to Solve:

$$G_n'(t) = -\lambda_n k\, G_n(t), \quad n = 0, 1, \ldots$$

First order linear ODE homogeneous

$$G_n(t) = F_n e^{-\lambda_n k t}, \quad n = 0, 1, \ldots$$

Therefore, we have the following $\boxed{\text{product Solutions}}$:

$\lambda_0 = 0$ $\quad u_0(x,t) = G_0(t)\,\phi_0(t) = a_0(1)$

$\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, \ldots$ $\quad u_n(x,t) = F_n e^{-\lambda_n k t}\left[c_n \cos\left(\frac{n\pi}{L}x\right) + d_n \sin\left(\frac{n\pi}{L}x\right)\right]$

or λ_n $_{n=1,2..}$: $u_n(x,t) = e^{-\left(\frac{n\pi}{L}\right)^2 k t}\left[a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right]$

Because the heat equation is a linear equation:

$$\boxed{u_N(x,t) = a_0 + \sum_{n=1}^{N} e^{-\left(\frac{n\pi}{L}\right)^2 k t}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)} \quad (5.1)$$

is also a Solution. This is called the principle of Superposition.

The superposition of product solutions (5.1) Satisfies (1.1), (1.2), and (1.3) but for a general $f(x)$, it doesn't Satisfy (1.4)

$$\boxed{u(x,0) = f(x)}$$

Since,
$$u_N(x,0) = a_0 + \sum_{n=1}^{N}\left[a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right] \neq f(x)$$
in general.

Fourier Series representation for $f(x)$ Sufficiently Smooth.

It will be proved later that for functions $f(x)$ Sufficiently smooth, the infinite series

$$\boxed{u(x,0) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right] = f(x)} \quad (5.2)$$

Converges to $f(x)$ in $[-L, L]$ if $f(x)$ is continuous.

where
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx \tag{6.1}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\sin\frac{n\pi}{L}x\,dx \tag{6.2}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,\cos\frac{n\pi}{L}x\,dx. \tag{6.3}$$

The coefficients (6.1) – (6.3) are determined from the following orthogonality relationships for $\sin\frac{n\pi}{L}x$, $\cos\frac{n\pi}{L}x$

$$\int_{-L}^{L}\cos\frac{n\pi}{L}x\,\cos\frac{m\pi}{L}x\,dx = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0 \\ 2L, & n=m=0 \end{cases}$$

$$\int_{-L}^{L}\sin\frac{n\pi}{L}x\,\sin\frac{m\pi}{L}x\,dx = \begin{cases} 0, & n\neq m \\ L, & n=m\neq 0 \end{cases}$$

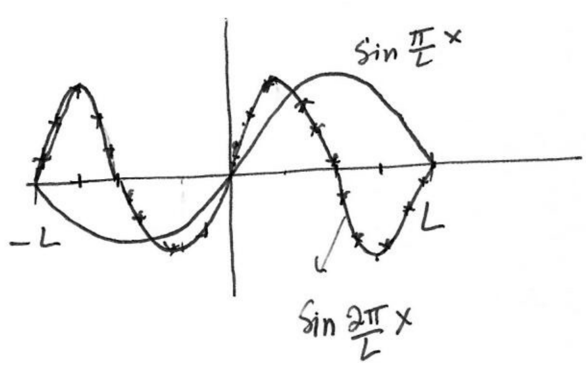
$$\int_{-L}^{L}\underbrace{\underset{\text{odd}}{\sin\frac{n\pi}{L}x}\;\underset{\text{even}}{\cos\frac{m\pi}{L}x}}_{\text{odd in a symmetric interval}}\,dx \overset{\text{Trivial}}{=} 0, \quad \text{for any } n, m \geq 0$$

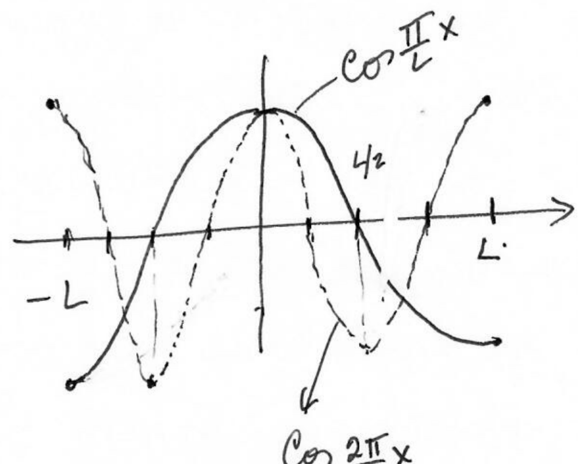
In fact,

multiplying by 1 $^{(5.2)}$, and $\int_{-L}^{L} dx$. and interchanging $\int_{-L}^{L} dx \to \sum_{n=1}^{\infty}$

$$\int_{-L}^{L} f(x)\,dx = A_0 \int_{-L}^{L} dx + \sum_{n=1}^{\infty} A_n \int_{-L}^{L} \cos \frac{n\pi}{L} x\, dx + $$

$$+ \sum_{n=1}^{\infty} B_n \int_{-L}^{L} \sin \frac{n\pi}{L} x\, dx$$





Clearly,

$$\int_{-L}^{L} \sin \frac{\pi}{L} x\, dx = 0 .$$ In general $n = 1, 2, \ldots$

$$\int_{-L}^{L} \sin \frac{n\pi}{L} x\, dx = 0$$

Clearly,

$$\int_{-L}^{L} \cos \frac{\pi}{L} x\, dx = 0$$

And in general

$$\int_{-L}^{L} \cos \frac{n\pi}{L} x\, dx = 0.$$

Notice that

$$\int_{-L}^{L} \cos\left(\frac{n\pi}{L} x\right) dx = \frac{L}{n\pi} \sin\left(\frac{n\pi}{L} x\right)\Big|_{-L}^{L} = \frac{L}{n\pi}\left[\cancelto{0}{\sin(n\pi)} - \cancelto{0}{\sin(-n\pi)}\right] = 0$$

Thus,

$$\int_{-L}^{L} f(x)\,dx = A_0\,(L-(-L)) = 2LA_0$$

$$\Rightarrow \boxed{A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx.}$$

Similarly, multiplying by $\sin\left(\frac{m\pi}{L}x\right)$, $m \geq 1$, $\int_{-L}^{L} dx$ and interchanging.

$$\int_{-L}^{L} f(x) \sin\frac{m\pi}{L}x\,dx =$$

$$= A_0 \overbrace{\int_{-L}^{L} \sin\frac{n\pi}{L}x\,dx}^{0} + \sum_{n=1}^{\infty} A_n \int_{-L}^{L} \sin\frac{m\pi}{L}x \sin\frac{n\pi}{L}x\,dx$$

$$+ \sum_{n=1}^{\infty} \overbrace{\int_{-L}^{L} \sin\frac{m\pi}{L}x \cos\frac{n\pi}{L}x\,dx}^{0 \text{ (form. 5.4)}} =$$

$$= A_n \int_{-L}^{L} \sin^2\left(\frac{n\pi}{L}x\right) dx \overset{\text{form (5.3)}}{=} A_n L.$$

$$\Rightarrow \boxed{A_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\frac{n\pi}{L}x\,dx.} \qquad n = 1, 2, \ldots$$

Analogously,

$$\boxed{B_n = \frac{1}{L}\int_{-L}^{L} f(x) \cos\frac{n\pi}{L}x\,dx. \qquad n = 1, 2 \ldots}$$

Previous Math knowledge used in the method of Separation of Variables.

① Finding Solns. for

a) 1^{st} order linear homogeneous equ.

b) 2^{nd} " " " "

② Finding eigenvalues and eigenvectors.

$$A\vec{x} = \lambda\vec{x}$$

$$L\phi = -\lambda\phi$$

$$\phi''(x) = -\lambda\phi(x).$$

③ Concept of linear operator

If $u_1, ., u_N$ are solutions

of $Lu = 0 \Rightarrow L(u_1 + \cdots + u_N) = L\left(\sum_{i=1}^{N} u_i\right) = 0$

④ Concept of orthogonality. Orthogonal vectors.

$\vec{u} \perp \vec{v}$ if and only if $\vec{u} \cdot \vec{v} = 0$

If Space of functions (Square integrable)

$f(x) \perp g(x)$ on $[-L, L]$ if

$$\int_{-L}^{L} f(x)g(x)\,dx = 0.$$

Here the function $f(x)$ is a linear combination of both sines and cosines (and a constant), unlike the previous problems, where either sines or cosines (including the constant term) were used. Another crucial difference is that (2.4.39) should be valid for the entire ring, which means that $-L \le x \le L$, whereas the series of just sines or cosines was valid for $0 \le x \le L$. The theory of Fourier series will show that (2.4.39) is valid and, more important, that the previous series of just sines or cosines are but special cases of the series in (2.4.39).

For now, we wish just to determine the coefficients a_0, a_n, and b_n from (2.4.39). Again the eigenfunctions form an orthogonal set since integral tables verify the following orthogonality conditions:

$$\boxed{\int_{-L}^{L} \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \\ 2L & n = m = 0 \end{cases}} \tag{2.4.40}$$

$$\boxed{\int_{-L}^{L} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ L & n = m \neq 0 \end{cases}} \tag{2.4.41}$$

$$\boxed{\int_{-L}^{L} \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0.} \tag{2.4.42}$$

where n and m are arbitrary (nonnegative) integers. The constant eigenfunction corresponds to $n = 0$ or $m = 0$. Integrals of the square of sines or cosines ($n = m$) are evaluated again by the "half the length of the interval" rule. The last of these formulas, (2.4.42), is particularly simple to derive, since sine is an odd function and cosine is an even function.[4] Note that, for example, $\cos n\pi x/L$ is orthogonal to every *other* eigenfunction [sines from (2.4.42), cosines and the constant eigenfunction from (2.4.40)].

The coefficients are derived in the same manner as before. A few steps are saved by noting (2.4.39) is equivalent to

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}.$$

If we multiply this by both $\cos m\pi x/L$ and $\sin m\pi x/L$ and then integrate *from* $x = -L$ to $x = +L$, we obtain

[4] The product of an odd and an even function is odd. By antisymmetry, the integral of an odd function over a symmetric interval is zero.

$x = -L$ to $x = +L$, we obtain

$$\int_{-L}^{L} f(x) \left\{ \begin{array}{c} \cos \dfrac{m\pi x}{L} \\ \sin \dfrac{m\pi x}{L} \end{array} \right\} dx = \sum_{n=0}^{\infty} a_n \int_{-L}^{L} \cos \frac{n\pi x}{L} \left\{ \begin{array}{c} \cos \dfrac{m\pi x}{L} \\ \sin \dfrac{m\pi x}{L} \end{array} \right\} dx$$
$$+ \sum_{n=1}^{\infty} b_n \int_{-L}^{L} \sin \frac{n\pi x}{L} \left\{ \begin{array}{c} \cos \dfrac{m\pi x}{L} \\ \sin \dfrac{m\pi x}{L} \end{array} \right\} dx.$$

If we utilize (2.4.40–2.4.42), we find that

$$\int_{-L}^{L} f(x) \cos \frac{m\pi x}{L}\, dx = a_m \int_{-L}^{L} \cos^2 \frac{m\pi x}{L}\, dx$$

$$\int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\, dx = b_m \int_{-L}^{L} \sin^2 \frac{m\pi x}{L}\, dx.$$

Solving for the coefficients in a manner that we are now familiar with yields

$$\boxed{\begin{array}{rrcl} & a_0 & = & \dfrac{1}{2L} \displaystyle\int_{-L}^{L} f(x)\, dx \\ (m \geq 1) & a_m & = & \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \cos \frac{m\pi x}{L}\, dx \\ & b_m & = & \dfrac{1}{L} \displaystyle\int_{-L}^{L} f(x) \sin \frac{m\pi x}{L}\, dx. \end{array}} \quad (2.4.43)$$

The solution to the problem is (2.4.38), where the coefficients are given by (2.4.43).

2.4.3 Summary of Boundary Value Problems

In many problems, including the ones we have just discussed, the specific simple constant-coefficient differential equation,

$$\frac{d^2\phi}{dx^2} = -\lambda\phi,$$

forms the fundamental part of the boundary value problem. We collect in the table in one place the relevant formulas for the eigenvalues and eigenfunctions for the typical boundary conditions already discussed. You will find it helpful to understand these results because of their enormous applicability throughout this text. It is important to note that, in these cases, whenever $\lambda = 0$ is an eigenvalue, a constant is the eigenfunction (corresponding to $n = 0$ in $\cos n\pi x/L$).

Table 2.4.1: Boundary Value Problems for $\dfrac{d^2\phi}{dx^2} = -\lambda\phi$

Boundary conditions	$\phi(0) = 0$ $\phi(L) = 0$	$\dfrac{d\phi}{dx}(0) = 0$ $\dfrac{d\phi}{dx}(L) = 0$	$\phi(-L) = \phi(L)$ $\dfrac{d\phi}{dx}(-L) = \dfrac{d\phi}{dx}(L)$
Eigenvalues λ_n	$\left(\dfrac{n\pi}{L}\right)^2$ $n = 1, 2, 3, \ldots$	$\left(\dfrac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$	$\left(\dfrac{n\pi}{L}\right)^2$ $n = 0, 1, 2, 3, \ldots$
Eigenfunctions	$\sin\dfrac{n\pi x}{L}$	$\cos\dfrac{n\pi x}{L}$	$\sin\dfrac{n\pi x}{L}$ and $\cos\dfrac{n\pi x}{L}$
Series	$f(x) = \sum_{n=1}^{\infty} B_n \sin\dfrac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} A_n \cos\dfrac{n\pi x}{L}$	$f(x) = \sum_{n=0}^{\infty} a_n \cos\dfrac{n\pi x}{L}$ $+\sum_{n=1}^{\infty} b_n \sin\dfrac{n\pi x}{L}$
Coefficients	$B_n = \dfrac{2}{L}\int_0^L f(x)\sin\dfrac{n\pi x}{L}\,dx$	$A_0 = \dfrac{1}{L}\int_0^L f(x)\,dx$ $A_n = \dfrac{2}{L}\int_0^L f(x)\cos\dfrac{n\pi x}{L}\,dx$	$a_0 = \dfrac{1}{2L}\int_{-L}^L f(x)\,dx$ $a_n = \dfrac{1}{L}\int_{-L}^L f(x)\cos\dfrac{n\pi x}{L}\,dx$ $b_n = \dfrac{1}{L}\int_{-L}^L f(x)\sin\dfrac{n\pi x}{L}\,dx$

EXERCISES 2.4

*2.4.1. Solve the heat equation $\partial u/\partial t = k\partial^2 u/\partial x^2$, $0 < x < L$, $t > 0$, subject to

$$\frac{\partial u}{\partial x}(0,t) = 0 \qquad t > 0$$
$$\frac{\partial u}{\partial x}(L,t) = 0 \qquad t > 0.$$

(a) $u(x,0) = \begin{cases} 0 & x < L/2 \\ 1 & x > L/2 \end{cases}$ (b) $u(x,0) = 6 + 4\cos\dfrac{3\pi x}{L}$

(c) $u(x,0) = -2\sin\dfrac{\pi x}{L}$ (d) $u(x,0) = -3\cos\dfrac{8\pi x}{L}$